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# Long wave generation on a sloping beach

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A general solution of the linear long-wave equation is obtained for arbitrary ground motion on a uniformly sloping beach. Numerical results are presented for specific shapes and time histories of ground motion. Near-shore large amplitude waves are also investigated using non-linear theory.

## 1. Introduction

The present paper concerns two-dimensional wave generation due to bottom motion in shallow water where the undisturbed bottom surface consists of a uniformly sloping beach. The main application is to the study of tsunami waves, which are generally due to seismic disturbances.

Most previous analytical work on this subject has been restricted to constant depth situations, e.g. Kajiura (1963), Momoi (1964), but with the advantage of inclusion of dispersion, which is neglected here. The shallow-water (non-dispersive) equations have been solved numerically for some general three-dimensional bottom topographies by Aida (1969) and by Hwang & Divoky (1970). An exact solution of the non-linear shallow-water equations for run-up onto a sloping beach was given by Carrier & Greenspan (1958), and Carrier (1966) matched this solution to a linear dispersing incoming wave generated in a region of constant depth.

The purpose of the present work is to provide analytical solutions and qualitative discussion for the case when bottom motion occurs at a place where the bottom is actually sloping, the resulting wave then propagating away into deeper water. This is a situation somewhat closer to common seismic tsunami generating mechanisms. Dispersion is neglected in the generation region, where typical horizontal length scales are supposed to be much greater than the local water depth.

In fact we show that dispersion may still be neglected some distance away from the generation region, even though the depth is continually increasing. This is because significant wavelengths of the generated wave remain long compared to the water depth up to a distance of order $b/\alpha^2$, where $b$ is the scale of the ground motion and $\alpha$ the bottom slope. Beyond this distance dispersion is undoubtedly significant.

In the range $b \ll x \ll b/\alpha^2$ we can make use of an asymptotic result for the wave elevation. The amplitude of the generated wave in this 'intermediate' far-field zone is characterized by a relatively simple relationship with the space and time history of the ground motion.

Using this relationship we investigate the effects of transients in the ground motion, establishing a connexion between the time scale of the transient and the amplitude of the generated wave. The results for a particular class of transients confirm quantitatively that transients with the usual seismic time scales (seconds) generate negligible waves compared with those generated by step-function-like ground motions. This result is of some practical importance, since the detailed time history of an earthquake is quite difficult to estimate, whereas the permanent stepwise ground displacement is usually known and can be used in general numerical studies such as that of Hwang & Divoky (1970).

The foregoing conclusions are obtained from a linearized or small amplitude theory. Since this linearization may be questionable very close to the shore-line we also provide non-linear computations similar to those of Carrier & Greenspan (1958). In fact we establish a direct correspondence between a class of non-linear and a class of linear solutions, so that the previous linear results may be re-interpreted directly as providing solutions of the non-linear problem with modified initial conditions.

## 2. Shallow-water equations with bottom motion

The equations of shallow-water or long-wave theory are well known. However, these equations are usually derived on the assumption that the bottom is non-moving. The additional contributions due to bottom motion are easy to establish, either by physical arguments or by careful re-derivation of the asymptotic expansions in the manner of Friedrichs (1948).

The resulting two-dimensional non-linear shallow-water equations are

$$\frac{\partial}{\partial t}(\eta+h)+\frac{\partial}{\partial x}u(\eta+h) = 0 \tag{2.1}$$

and

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}. \tag{2.2}$$

In these equations $u(x,t)$ is the horizontal velocity component, the bottom is given by

$$y = -h(x,t), \tag{2.3}$$

and the free surface by

$$y = \eta(x,t), \tag{2.4}$$

in the co-ordinate system indicated by figure 1. The effect of bottom motion is through the time dependence of $h$ in (2.1); if $h$ is independent of $t$, (2.1) reduces to the usual shallow-water equation.

If we suppose that an upward bottom displacement of magnitude $\eta_0(x,t)$ occurs where the undisturbed bottom shape is $y = -h_0(x)$, we may write

$$h(x,t) = h_0(x)-\eta_0(x,t), \tag{2.5}$$

in which case (2.1) becomes

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\{u(h_0+\eta)\} = \frac{\partial \eta_0}{\partial t}+\frac{\partial}{\partial x}(u\eta_0). \tag{2.6}$$

The advantage of (2.6) is that the forcing terms due to $\eta_0$ are shown separately.

Equations (2.2) and (2.6) describe waves of arbitrary amplitude. If we are prepared to restrict attention to waves of 'small' amplitude, we may linearize these

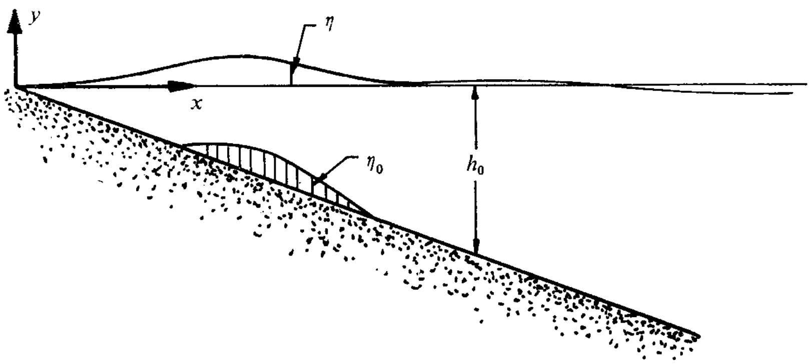



FIGURE 1. Schematic drawing of the ground motion and symbol definitions.

equations by assuming that $\eta$ and $\eta_0$ are small compared with $h_0$ and that $u$ is likewise small compared with the local wave speed $(gh_0)^{\frac{1}{2}}$. Neglecting second-order terms in (2.2) and (2.6) gives the linearized shallow-water equations

$$\frac{\partial u}{\partial t}+g\frac{\partial \eta}{\partial x} = 0 \tag{2.7}$$

and

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}(uh_0) = \frac{\partial \eta_0}{\partial t}. \tag{2.8}$$

Again, these equations are well known, apart from the bottom motion forcing term on the right of (2.8).

## 3. The solution for arbitrary small ground motion on a uniformly sloping beach

In the special case of a beach of uniform slope $\alpha$ we have

$$h_0(x) = \alpha x, \tag{3.1}$$

and (2.8) reduces to

$$\frac{\partial \eta}{\partial t}+\alpha\frac{\partial}{\partial x}(xu) = \frac{\partial \eta_0}{\partial t}. \tag{3.2}$$

The pair of linear equations (2.7), (3.2) may now be solved for quite arbitrary $\eta_0(x,t)$ by elementary Laplace and Hankel transformations, with the result

$$\eta(x,t) = \frac{2}{(g\alpha)^{\frac{1}{2}}}\int_0^\infty J_0(2\kappa(x)^{\frac{1}{2}})\,d\kappa\int_0^\infty J_0(2\kappa(\xi)^{\frac{1}{2}})\,d\xi\int_0^t \eta_1(\xi,\tau)\sin[(g\alpha)^{\frac{1}{2}}\kappa(t-\tau)]\,d\tau, \tag{3.3}$$

where $J_0$ is a Bessel function of the first kind, order zero.

The function $\eta_1(x,t)$ whose second time derivative appears in (3.3) is defined as

$$\eta_1(x,t) = \eta_0(x,t) + [\eta(x,0_-) + t\dot{\eta}(x,0_-)]\,H(t), \tag{3.4}$$

where $H(t)$ is a Heaviside unit step function. Thus $\eta_1(x,t)$ is a 'modified' ground motion, taking into account effects of the initial elevation $\eta(x,0_-)$ or velocity $\dot{\eta}(x,0_-)$ of the free surface at the instant before the ground begins to move.

For example, if there is no ground motion at all, i.e. $\eta_0 = 0$ for all $x$ and $t$, and if the initial velocity is zero, i.e. $\dot{\eta}(x,0_-) = 0$, then

$$\ddot{\eta}_1(x,t) = \eta(x,0_-)\,\dot{\delta}(t) \tag{3.5}$$

and

$$\int_0^t \ddot{\eta}_1(\xi,\tau)\sin[(g\alpha)^{\frac{1}{2}}\kappa(t-\tau)]\,d\tau = \eta(\xi,0_-)\,(g\alpha)^{\frac{1}{2}}\kappa\cos((g\alpha)^{\frac{1}{2}}\kappa t). \tag{3.6}$$

Thus the waves following an initial free-surface elevation $\eta(x,0_-)$ are given by

$$\eta(x,t) = 2\int_0^\infty J_0(2\kappa(x)^{\frac{1}{2}})\,\kappa\cos((g\alpha)^{\frac{1}{2}}\kappa t)\,d\kappa\int_0^\infty J_0(2\kappa(\xi)^{\frac{1}{2}})\,\eta(\xi,0_-)\,d\xi. \tag{3.7}$$

On the other hand, a step-function ground motion with an initially undisturbed fluid can lead to a wave identical to that for an initial elevation, i.e.

$$\eta_0(x,t) = \eta_0(x,0_+)\,H(t) \tag{3.8}$$

and

$$\eta(x,0_-) = \dot{\eta}(x,0_-) = 0 \tag{3.9}$$

imply

$$\ddot{\eta}_1(x,t) = \eta_0(x,0_+)\,\dot{\delta}(t), \tag{3.10}$$

and (3.7) still holds with $\eta_0(x,0_+)$ instead of $\eta(x,0_-)$. In this case, we find $\eta(x,0_+) = \eta_0(x,0_+)$, so that if the ground motion does begin with an upward step, that step appears instantaneously as a free-surface elevation, at $t = 0_+$, and is added to whatever initial elevation was present at $t = 0_-$. Of course in practice no ground motion begins quite so suddenly, but it is useful to be able to interpret a rapid ground motion of a step-function character as equivalent to an initial elevation of the free surface.

Specializing further, we may consider an initial elevation (or step-function ground motion) which decays exponentially away from the shore, writing

$$\eta(x,0_-) = a e^{-x/b}, \tag{3.11}$$

where $a$ is the elevation at the shoreline $x = 0$ and $b$ a measure of the horizontal extent of the initial disturbance. It is of course inconsistent with the linearization to allow a non-zero elevation $a$ at the shoreline, where the depth is zero, and this question is considered again in §6.

On substituting (3.11) in (3.7) we obtain

$$\eta(x,t) = 2ab\int_0^\infty \kappa e^{-b\kappa^2} J_0(2\kappa(x)^{\frac{1}{2}})\cos((g\alpha)^{\frac{1}{2}}\kappa t)\,d\kappa. \tag{3.12}$$

Figure 2 shows values of $\eta$ computed by numerical integration of (3.12) as a function of $x/b$, for various values of non-dimensional time $t' = t\,(g\alpha/b)^{\frac{1}{2}}$. Similarly figure 3 shows $\eta$ as a function of $t'$ for various (large) values of $x/b$. In fact, the

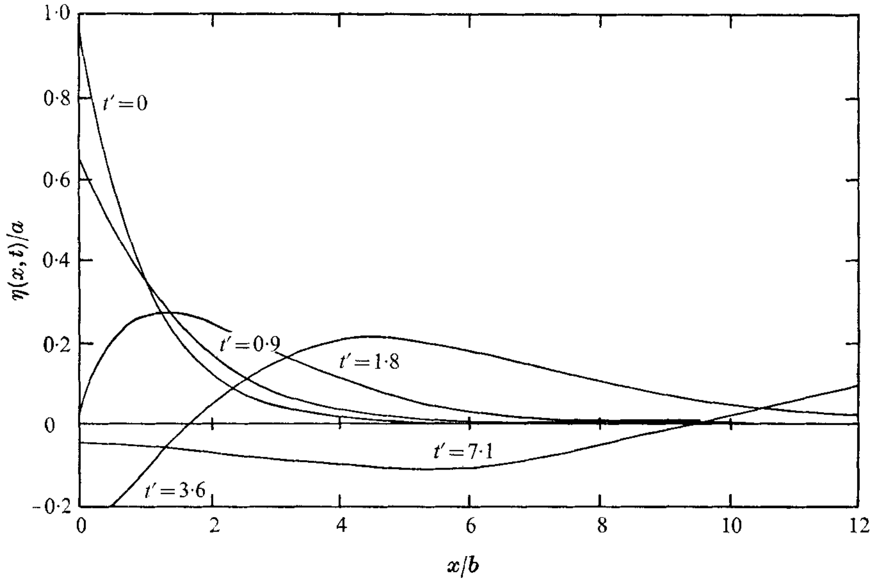



FIGURE 2. Successive variation of water surface elevation following an initial elevation or step ground motion with exponential $x$ dependence.

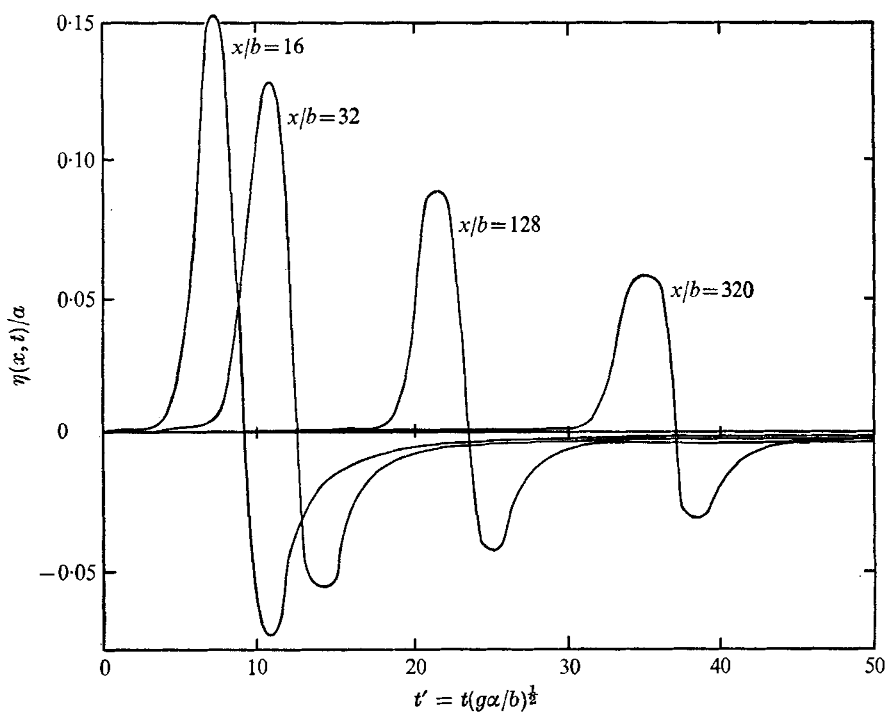



FIGURE 3. Offshore water surface elevation recorded as a function of time at a given location $x$; same initial conditions as figure 2.

shape of the curves in figure 3 is independent of $x$, a phenomenon which is further investigated in the following section. Similar computations have been carried out for non-exponential $x$ dependence.

## 4. The intermediate far-field waves

We now suppose that some kind of disturbance creates a wave in the neighbourhood of the shoreline $x = 0$ and that this disturbance is essentially completed after a finite time. The result will be a wave with a beginning and an end, travelling to $x = +\infty$. At a very great distance, since the water depth is continually increasing, we certainly expect dispersion effects to become important; however, we shall investigate here an 'intermediate' far field in which the shallow-water assumption is retained. The limits of validity of this assumption will be obtained as part of the analysis.

The appropriate asymptotic expansion is one which follows the wave, i.e. one in which $x$ and $t$ become large together. Upon replacement of the Bessel function $J_0(2\kappa(x)^{\frac{1}{2}})$ in (3.3) by its large amplitude asymptotic expansion we observe that unless the quantity

$$T = t - 2(x/g\alpha)^{\frac{1}{2}} \tag{4.1}$$

remains bounded as $t$ and $x$ tend to infinity, the integral with respect to $\kappa$ contains a highly oscillatory integrand and the wave elevation tends to zero rapidly. On the other hand, for bounded $T$ we have as $x, t \to \infty$ that

$$\eta(x,t) \to \frac{x^{-\frac{1}{4}}}{(\pi\alpha g)^{\frac{1}{2}}} \int_0^\infty d\xi \int_0^\infty \ddot{\eta}_1(\xi,\tau)\, d\tau \int_0^\infty \kappa^{-\frac{1}{2}} J_0(2\kappa(\xi)^{\frac{1}{2}}) \sin[(g\alpha)^{\frac{1}{2}}\kappa(T-\tau) + \tfrac{1}{4}\pi]\, d\kappa. \tag{4.2}$$

The wave described by (4.2) propagates towards $x = +\infty$ according to the equation

$$x = \tfrac{1}{4} g\alpha t^2, \tag{4.3}$$

i.e. it moves with constant acceleration $\frac{1}{2}g\alpha$. This is because, if we choose a relative time scale $T$ as defined in (4.1) such that $T = 0$ at $x = \frac{1}{4}g\alpha t^2$, the shape of the far-field wave is independent of $x$ and $t$, depending only on relative time $T$. For example, the highest point on the wave remains at the same value of $T$ for all time and the whole wave occupies an effectively finite and constant number of units of $T$.

The height of the wave decreases with time or distance from the source, according to the factor $x^{-\frac{1}{4}}$ (which is equivalent to $t^{-\frac{1}{2}}$) outside the integral (4.2). Since the depth increases as $x$ this corresponds to Green's law for shallow-water waves. For example, suppose the modified ground acceleration is given by (3.5) subject to (3.11). Then either by substitution in (4.2) or by direct asymptotic approximation of (3.12) we obtain

$$\eta(x,t) \to \frac{abx^{-\frac{1}{4}}}{(\pi)^{\frac{1}{2}}} \int_0^\infty d\kappa\kappa^{\frac{1}{2}} e^{-b\kappa^2} \cos((g\alpha)^{\frac{1}{2}}\kappa T + \tfrac{1}{4}\pi), \tag{4.4}$$

which gives wave shapes precisely as shown in figure 3, where $T$ is determined by (4.1). In particular, the elevation at $T = 0$ is seen to be

$$\eta(x, 2(x/g\alpha)^{\frac{1}{2}}) = a\left(\frac{\Gamma(\frac{3}{4})}{2(2\pi)^{\frac{1}{2}}}\right)(x/b)^{-\frac{1}{4}}$$
$$= 0{\cdot}246\, a(x/b)^{-\frac{1}{4}}. \quad (4.5)$$

The instant $t = 2(x/g\alpha)^{\frac{1}{2}}$ is only slightly later than the time of maximum positive elevation, so that the formula (4.5) gives a reasonable estimate of the size of the outgoing wave.

Finally, let us consider the validity of the neglect of dispersion in this section. Certainly we can expect that in practice if we go far enough away from the source of disturbance dispersion effects will play an increasingly important role, since the depth is becoming larger all the time. However, it appears that there is an intermediate zone, far from the source, but not yet far enough for dispersion to be significant.

The size of this intermediate region can be established as follows. Clearly we require $x \gg b$, where $b$ is, as in (3.11), a measure of the extent of the disturbance. Dispersion will be significant when important wavelengths are comparable with the water depth. However it is clear from (4.2) and figure 3 that the wave is spreading out as $x$ or $t$ increases in such a way that it occupies a distance of the order of $(bx)^{\frac{1}{2}}$ at each fixed (large) value of $t$. Thus the ratio between (significant) wavelength and water depth is of the order of $(bx)^{\frac{1}{2}}/\alpha x$, which becomes of the order of unity when $x$ is of the order of $b/\alpha^2$.

Thus the region of validity of the asymptotic analysis of the present section is

$$b \ll x \ll b/\alpha^2. \quad (4.6)$$

Note that this applies only to the main part of the wave and that dispersion will become significant earlier than $x \sim b/\alpha^2$ for some of the higher frequency components. The intermediate far-field expansion is therefore useful as a 'figure of merit' indication of tsunami generation, it being left for more detailed finite depth analysis to describe the subsequent propagation and dispersion of the wave.

## 5. Transient ground motion

One problem in attempting to study generation of tsunamis is that in practice the detailed time history of ground motion due to an earthquake is generally unknown. Without this information, the best that can be done (see e.g. Hwang & Divoky 1970) is to use a step equal to the permanent deformation, or the net change in ground level from before the shock to after the shock, a quantity which can be measured (Plafker 1969). This assumption appears to be justified on the basis that the detailed time history of the shock, occurring as it does over periods of seconds, is unimportant as a generator of long period (hours) tsunami waves.

We may test this assumption quantitatively using the present model theory by considering a transient ground motion of the form

$$\eta_1(x,t) = \eta_0(x,t) = a\gamma^2 t^2\, e^{-\gamma t}\, e^{-x/b} \quad (5.1)$$

which begins slowly (zero initial displacement and velocity), reaches a maximum of $0{\cdot}54a$ at $t = 2/\gamma$ and then decays exponentially to its original level. The time scale of the transient is measured by $1/\gamma$; for instance the whole ground motion is essentially completed within a time (say) $t = 10/\gamma$. The expression (5.1) is quite arbitrary and is chosen because the resulting equations are tractable and it does represent a transient rise and fall which, for proper choice of $\gamma$, should not be too unlike the real situation.

On substitution of the expression (5.1) for the modified ground motion $\eta_1(x, t)$ into the general formula (3.3) we obtain a complicated expression including transient wave effects which decay in time and space at the same rate as the transient ground motion, together with outgoing wave terms which decay much more slowly and represent the generated tsunami. At distances from the earthquake satisfying the inequality (4.6) these latter terms are adequately described by the intermediate far-field asymptotic theory of the previous section.

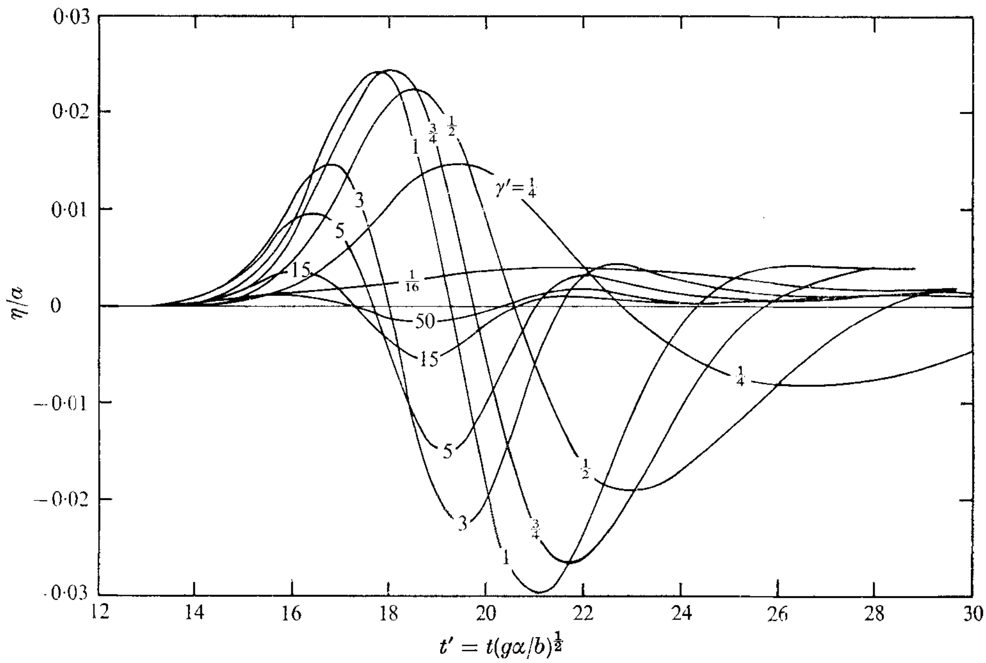



FIGURE 4. The dependence of wave amplitude on the transient parameter $\gamma' = \gamma(b/g\alpha)^{\frac{1}{2}}$ at $x/b = 80$.

Thus from (4.2) we find that the transient ground motion (5.1) leads to a tsunami of the form

$$\eta(x, t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} abx^{-\frac{1}{4}} \int_0^\infty \kappa^{-\frac{1}{2}} e^{-b\kappa^2} [A\cos\theta + B\sin\theta]\, d\kappa, \tag{5.2}$$

where
$$\theta = (g\alpha)^{\frac{1}{2}} \kappa T + \tfrac{1}{4}\pi, \tag{5.3}$$

$$A = -g\alpha\gamma^2\kappa^3(g\alpha\kappa^2 - 3\gamma^2)(g\alpha\kappa^2 + \gamma^2)^{-3} \tag{5.4}$$

and
$$B = (g\alpha)^{\frac{1}{2}} \kappa^2\gamma^3(3g\alpha\kappa^2 - \gamma^2)(g\alpha\kappa^2 + \gamma^2)^{-3}, \tag{5.5}$$

$T$ being given by (4.1). Wave elevations at various values of $\gamma' = \gamma(b/g\alpha)^{\frac{1}{2}}$ are shown in figure 4 for the station $x/b = 80$.

One interesting feature of equation (5.2) and figure 4 is that the wave elevation vanishes both as $\gamma \to 0$ and as $\gamma \to \infty$. Thus, as $\gamma \to 0$, $A \sim \gamma^2/\kappa$ and $B \sim \gamma^3/\kappa^2$, so that $\eta \to 0$ like $\gamma^2$. On the other hand, as $\gamma \to \infty$, $A \sim \kappa^3/\gamma^2$ and $B \sim \kappa^2/\gamma$ so that $\eta \to 0$ like $1/\gamma$. The physical explanation for this effect is that as $\gamma \to 0$ the ground motion is of the nature of a very slow upwelling over a period of (say) days, in which case the tsunami generation will be negligible, whereas for $\gamma \to \infty$ we obtain the true earthquake situation in which the transient ground motion periods may be of the order of seconds and, again, negligible tsunami production occurs owing to these fast transients. At intermediate values of $\gamma$ (about $\gamma(b/g\alpha)^{\frac{1}{2}} = 1$) the tsunami magnitude reaches a maximum value, corresponding to peak efficiency of generation by this transient ground motion. The corresponding decay time (about $10\,(b/g\alpha)^{\frac{1}{2}}$) of this 'optimum' transient ground motion is however much longer than typical time scales for transient ground motion due to an earthquake, which correspond to $\gamma(b/g\alpha)^{\frac{1}{2}}$ of fifty or more.

The conclusion to be drawn from these results is that a ground motion which lasts only a brief time and results in little permanent deformation will indeed generate a negligible wave. Since the present analysis is linear, this also implies that when permanent deformation is present the wave generated can be computed from this permanent deformation, ignoring rapid transients. Although this conclusion is intuitively natural, it is desirable to have checks such as the present results on its validity. More detailed information about transients can be obtained by considering the 'frequency response', i.e. the steady-state amplitude of wave generation by a *periodic* ground motion; however, the same conclusion should be obtained.

## 6. Non-linear solution for an initial elevation

If there is no ground motion at all and the bottom is one of constant slope $\alpha$, we have from (2.6) and (3.1) that

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} u(\alpha x + \eta) = 0. \qquad (6.1)$$

It is remarkable that the pair of thoroughly non-linear equations (2.2), (6.1) can in this case be converted to a pair of linear equations by an appropriate change of variables. Such a transformation was inferred from results of Stoker (1948) by Carrier & Greenspan (1958).

We use the following new independent variables:

$$x^* = x + \eta/\alpha \qquad (6.2)$$

and

$$t^* = t - u/g\alpha, \qquad (6.3)$$

with dependent variables

$$u^*(x^*, t^*) = u(x, t) \qquad (6.4)$$

and

$$\eta^*(x^*, t^*) = \eta + u^2/2g. \qquad (6.5)$$

Thus $u^*$ is identical to the velocity $u$ but is expressed in terms of the starred coordinates, whereas $\eta^*$ differs from the surface elevation $\eta$ by the quantity $u^2/2g$.

When equations (6.2)–(6.5) are used in conjunction with (2.2) and (6.1) we obtain the linear equations

$$\frac{\partial u^*}{\partial t^*}+g\frac{\partial \eta^*}{\partial x^*} = 0, \tag{6.6}$$

and

$$\frac{\partial \eta^*}{\partial t^*}+\alpha\frac{\partial}{\partial x^*}(x^*u^*) = 0. \tag{6.7}$$

Thus, not only has this transformation linearized the governing equations, but it has converted them into precisely the linear equations (2.7) and (3.2) which describe small amplitude waves. The transformation (6.2)–(6.5) is, with small modifications, equivalent to that of Carrier & Greenspan (1958), but the latter does not possess the above feature.

Since (6.6) and (6.7) have already been solved in §3 we need merely replace unstarred co-ordinates by starred co-ordinates to obtain solutions for $\eta^*$, and hence $u^*$, as functions of $x^*$ and $t^*$. Equations (6.2)–(6.5) then determine $\eta$ and $u$ implicitly as functions of $x$ and $t$.

For example, (3.12) re-written in starred co-ordinates states

$$\eta^*(x^*,t^*) = 2ab\int_0^\infty \kappa e^{-b\kappa^2} J_0(2\kappa(x^*)^{\frac{1}{2}})\cos((g\alpha)^{\frac{1}{2}}\kappa t^*)\,d\kappa, \tag{6.8}$$

from which (6.6) provides the corresponding velocity

$$u^*(x^*,t^*) = ab\left(\frac{g}{\alpha x^*}\right)^{\frac{1}{2}}\int_0^\infty e^{-b\kappa^2} J_1(2\kappa(x^*)^{\frac{1}{2}})\sin((g\alpha^{\frac{1}{2}})\kappa t^*)\,d\kappa. \tag{6.9}$$

This is the solution with initial conditions such that

$$\eta^*(x^*,0_-) = a\,e^{-x^*/b} \tag{6.10}$$

and

$$u^*(x^*,0_-) = 0. \tag{6.11}$$

Since tabulated values of $\eta^*(x^*,t^*)$ and $u^*(x^*,t^*)$ are already available from the small amplitude results plotted in figure 2 we can use (6.2) and (6.8) to determine $x = x(x^*,t^*)$, and (6.3) and (6.9) to determine $t = t(x^*,t^*)$. Equations (6.4) and (6.5) then determine $u(x,t)$ and $\eta(x,t)$.

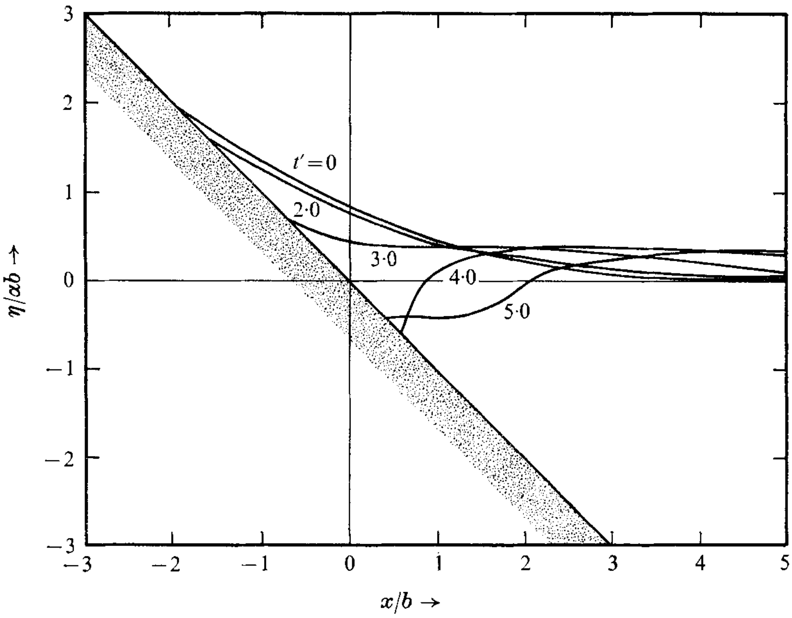
It should be noted that (6.10) does not give the actual initial elevation in this case. Equations (6.11) and (6.3) do guarantee that $t^* = 0$ corresponds to $t = 0$, and (6.5) indicates that at this instant $\eta^* = \eta$; however it is not true that $x^* = x$, and in fact we have from (6.2) and (6.10) that at $t = 0$,

$$x = x^* - a\,e^{-x^*/b}/\alpha. \tag{6.12}$$

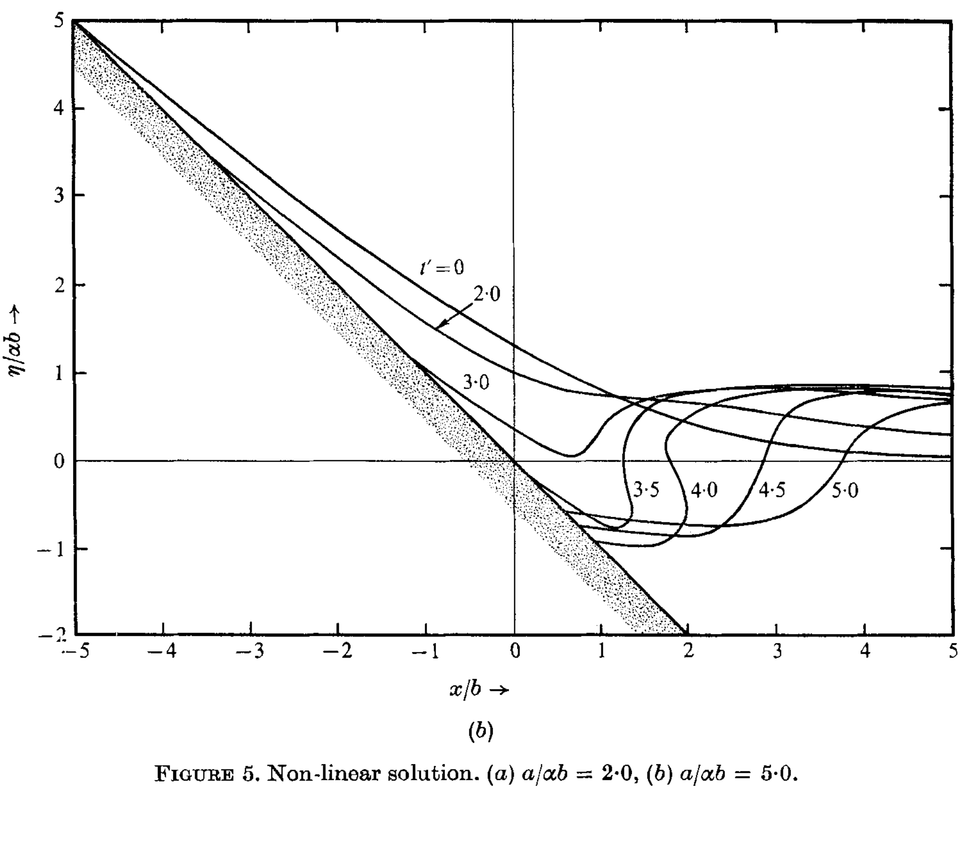
On solving (6.10) for $x^*$ we find that $\eta = \eta\,(x,0_-)$ is a solution of the transcendental equation

$$x = -b\log(\eta/a)-\eta/\alpha. \tag{6.13}$$

The initial elevation predicted by (6.13) is shown in figure 5 (as the curve labelled $t' = 0$) for the cases $a = 2\alpha b$ and $a = 5\alpha b$. Note that as $a \to 0$ we retrieve the linearized result that $x^* \to x$ and $\eta \to a\,e^{-x/b}$. On the other hand, the initial elevation given by (6.13) is physically more satisfactory since its extreme left edge is at $x = -a/\alpha$, where $\eta = a$. This point $(-a/\alpha, a)$ lies on the beach to the left




(*a*)




(*b*)

FIGURE 5. Non-linear solution. (*a*) $a/\alpha b = 2{\cdot}0$, (*b*) $a/\alpha b = 5{\cdot}0$.

of the 'equilibrium' shoreline $x = 0$, whereas the linearized solution requires a piling up of water at $x = 0$.

Figure 5 shows the solution for $\eta(x,t)$ for the cases $a = 2\alpha b$ and $a = 5\alpha b$. These results are plotted non-dimensionally in the form of $\eta' = \eta/b\alpha$ against $x' = x/b$ for various values of $t' = t(g\alpha/b)^{\frac{1}{2}}$. Note that in the very substantial initial disturbance $a = 5\alpha b$ of figure 5(*b*) the wave appears to almost break back upon itself at about $t' = 3{\cdot}5$.

Indeed we really have no right to use the present solution for such a severe disturbance, since it happens that the transformation between $(x^*, t^*)$ and $(x, t)$ fails to be one-to-one at some values of $x^*$ and $t^*$, and no doubt in practice actual breaking would occur. Even though there may exist more than one point $(x, t)$ for each $(x^*, t^*)$ it is still possible to pick out the set of continuously varying wave contours shown in figure 5(*b*); these are presented as mathematical curiosities only. A similar interpretation of 'post-breaking' calculations is given by Stoker (1948). A less severe breaking crisis occurs at $a = 2\gamma b$, and the results in figure 5(*a*) should have practical significance, although this case is very close to the borderline for breaking.

One remarkable feature of these results is that once the breaking crisis is past the surface immediately settles down and is flat near $x = 0$, with a wave spreading out to $x = +\infty$ in a manner similar to that of the linearized solution in figure 2. This property may be established analytically by observing from the intermediate far-field analysis of §4 that as $x^* \to \infty$, $\eta^* \to 0$ like $(x^*)^{-\frac{1}{4}}$, whereas $u^* \to 0$ like $(x^*)^{-\frac{3}{4}}$. Thus, ultimately $u^2/2g \ll \eta^*$ so that $\eta^* \to \eta$, $u/g\alpha \ll t^*$ so that $t^* \to t$, and $\eta/\alpha \ll x^*$ so that $x^* \to x$. Hence as $x \to \infty$ the linearized solution $\eta = \eta^*(x,t)$ is retrieved. We may therefore interpret the non-linear results as providing a justification for use of the linear theory even when we have reason to suspect its near-shore accuracy, providing we determine the actual initial elevation implicitly from the given 'linearized' initial elevation $\eta^*(x^*, 0_-)$.

Similar non-linear computations may be carried out for other specifications of $\eta^*(x^*, 0_-)$; some such results are given by Carrier & Greenspan (1958). The question of whether the wave will break can only be answered by computation of the Jacobian relating $(x^*, t^*)$ and $(x, t)$ for given $\eta^*(x^*, 0_-)$. In the present case, with $\eta^*(x^*, 0_-)$ given by (6.10), this Jacobian remains positive for $a/\alpha b$ less than about $2{\cdot}0$, so that the transformation remains one-to-one and no breaking occurs. It does not appear possible to provide an *a priori* breaking criterion independent of the actual specification of $\eta^*(x^*, 0_-)$.

It would, of course, be of great interest to compute non-linear solutions in the presence of actual ground motions $\eta_0(x,t)$. However, the basic transformation (6.2)–(6.5) does not appear to work in that case. Hence we must rely on the idea expressed in §3 that a ground motion of a step nature is equivalent to an initial elevation of the free surface. This equivalence is exact in the linearized case and may have qualitative significance in the non-linear case.

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